Fall 2018

MA 26600
Study Guide #2

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(1) **First Order Differential Equations.** (Separable, 1^{st} Order Linear, Homogeneous, Exact)

(2) **Second Order Linear Homogeneous with Equations Constant Coefficients** .

The differential equation $ay'' + by' + cy = 0$ has *Characteristic Equation* $ar^2 + br + c = 0$. Call the roots r_1 and r_2. The general solution of $ay'' + by' + cy = 0$ is as follows:

(a) If r_1, r_2 are real and distinct $\Rightarrow$ $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

(b) If $r_1 = \lambda + i\mu$ (hence $r_2 = \lambda - i\mu$) $\Rightarrow$ $y = C_1 e^{\lambda t}\cos\mu t + C_2 e^{\lambda t}\sin\mu t$

(c) If $r_1 = r_2$ (repeated roots) $\Rightarrow$ $y = C_1 e^{r_1 t} + C_2 t e^{r_1 t}$

(3) **Theory of 2^{nd} Linear Order Equations**.

Wronskian of y_1, y_2 is $W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$.

(a) The functions $y_1(t)$ and $y_2(t)$ are linearly independent over $a < t < b$ if $W(y_1, y_2) \neq 0$ for at least one point in the interval.

(b) **THEOREM (Existence & Uniqueness)** If $p(t), q(t)$ and $g(t)$ are continuous in an open interval $\alpha < t < \beta$ containing t_0, then the IVP $\begin{cases} y'' + p(t)\,y' + q(t)\,y = g(t) \\ y(t_0) = y_0 \\ y'(t_0) = y_1 \end{cases}$ has a unique solution $y = \phi(t)$ defined in the open interval $\alpha < t < \beta$.

(c) **Superposition Principle** If $y_1(t)$ and $y_2(t)$ are solutions of the 2^{nd} order linear homogeneous equation $P(t)y'' + Q(t)y' + R(t)y = 0$ over the interval $a < t < b$, then $y = C_1\,y_1(t) + C_2\,y_2(t)$ is also a solution for any constants C_1 and C_2.

(d) **THEOREM (Homogeneous)** If $y_1(t)$ and $y_2(t)$ are solutions of the linear homogeneous equation $P(t)y'' + Q(t)y' + R(t)y = 0$ in some interval I and $W(y_1, y_2) \neq 0$ for some t_1 in I, then the general solution is $y_c(t) = C_1\,y_1(t) + C_2\,y_2(t)$. This is usually called the *complementary solution* and we say that $y_1(t), y_2(t)$ form a *Fundamental Set of Solutions* (FSS) to the differential equation.

(e) **THEOREM (Nonhomogeneous)** The general solution of the nonhomogeneous equation

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

is $y(t) = y_c(t) + y_p(t)$, where $y_c(t) = C_1\,y_1(t) + C_2\,y_2(t)$ is the general solution of the corresponding homogeneous equation $P(t)y'' + Q(t)y' + R(t)y = 0$ and $y_p(t)$ is a particular solution of the nonhomogeneous equation $P(t)y'' + Q(t)y' + R(t)y = G(t)$.

(f) **Useful Remark** : If $y_{p_1}(t)$ is a particular solution of $P(t)y'' + Q(t)y' + R(t)y = G_1(t)$ and if $y_{p_2}(t)$ is a particular solution of $P(t)y'' + Q(t)y' + R(t)y = G_2(t)$, then

$$y_p(t) = y_{p_1}(t) + y_{p_2}(t)$$

is a particular solution of $P(t)y'' + Q(t)y' + R(t)y = [G_1(t) + G_2(t)]$.

(4) **Reduction of Order**. If $y_1(t)$ is one solution of $P(t)y'' + Q(t)y' + R(t)y = 0$, then a second solution may be obtained using the substitution $\boxed{\mathbf{y = v(t)\, y_1(t)}}$. This reduces the original 2^{nd} order equation to a 1^{st} equation using the substitution $\mathrm{w} = \dfrac{\mathrm{dv}}{\mathrm{dt}}$. Solve that first order equation for w, then since $\mathrm{w} = \dfrac{\mathrm{dv}}{\mathrm{dt}}$, solve this 1^{st} order equation to determine the function v.

(5) **Finding A Particular Solution $y_p(t)$ to Nonhomogeneous Equations**.

You can always use the method of Variation of Parameters to find a particular solution $y_p(t)$ of the linear nonhomogeneous equation $y'' + p(t)\,y' + q(t)\,y = g(t)$. Variation of Parameters may require integration techniques.

If the coefficients of the differential equation are constants rather than functions **and** if $g(t)$ has a very special form (see table below), it is usually easier to use Undetermined Coefficients :

(a) Undetermined Coefficients - **IF** $ay'' + by' + cy = g(t)$ **AND** $g(t)$ is as below:

$\boldsymbol{g(t)}$	Form of $\boldsymbol{y_p(t)}$
$P_m(t) = a_m t^m + a_{m-1}t^{m-1} + \cdots + a_0$	$t^s\,\{A_m t^m + A_{m-1}t^{m-1} + \cdots + A_0\}$
$e^{\alpha t}\,P_m(t)$	$t^s\,\{e^{\alpha t}\,(A_m t^m + A_{m-1}t^{m-1} + \cdots + A_0)\}$
$e^{\alpha t}\,P_m(t)\,\cos\beta t$ or $e^{\alpha t}\,P_m(t)\,\sin\beta t$	$t^s\,\{e^{\alpha t}\,[F_m(t)\,\cos\beta t + G_m(t)\,\sin\beta t]\}$

where s = the smallest nonnegative integer ($s = 0, 1$ or 2) such that no term of $y_p(t)$ is a solution of the corresponding homogeneous equation. In other words, no term of $y_p(t)$ is a term of $y_c(t)$. ($F_m(t)$, $G_m(t)$ are both polynomials of degree m.)

(b) Variation of Parameters - If $y_1(t)$ and $y_2(t)$ are two independent solutions of the homogeneous equation $y'' + p(t)\,y' + q(t)\,y = 0$, then a particular solution $y_p(t)$ of the nonhomogeneous equation

$$y'' + p(t)\,y' + q(t)\,y = g(t) \quad (*)$$

has the form

$$y_p(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$$

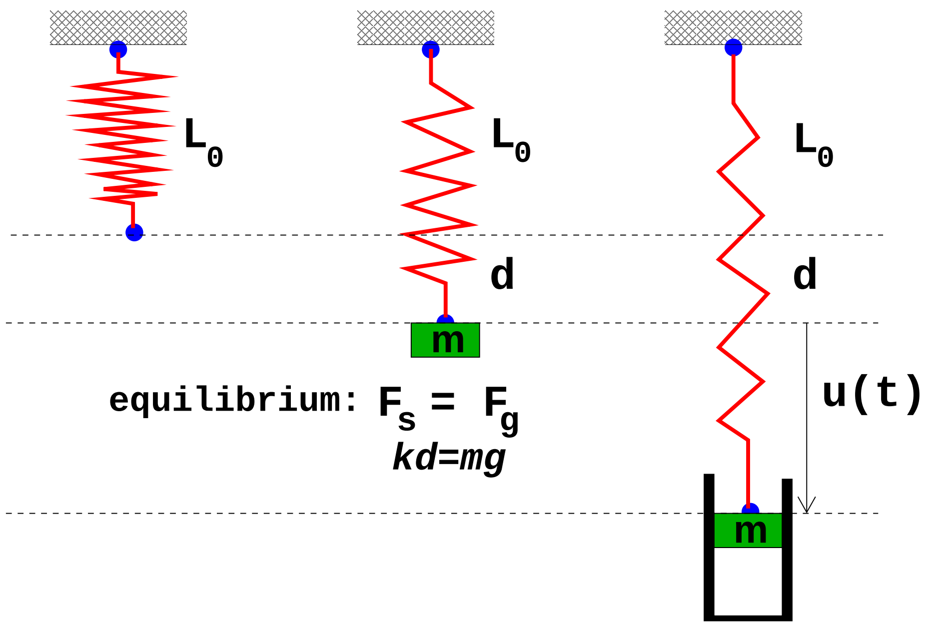
where

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g(t) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}, \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g(t) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}.$$

Remember: Coefficient of y'' in $(*)$ must be "1" in order to use the above formulas.

(6) Spring-Mass Systems $\begin{cases} m\,u'' + \gamma\,u' + k\,u = F(t) \\ u(0) = u_0\,,\;\; u'(0) = u_1 \end{cases}$

$m =$ mass of object, $\gamma =$ damping constant, $k =$ spring constant, $F(t) =$ external force Weight $w = m\,g$, Hooke's Law : $F_s = k\,d$,



$\boxed{\text{I}}$ Undamped Free Vibrations : $m\,u'' + k\,u = 0$ (Simple Harmonic Motion)
Note that $A\cos\omega_0 t + B\sin\omega_0 t = R\cos(\omega_0 t - \delta)$, where $R = \sqrt{A^2+B^2} =$ *amplitude*, $\omega_0 =$ *frequency*, $\dfrac{2\pi}{\omega_0} =$ *period* and $\delta =$ *phase shift* determined by $\tan\delta = \dfrac{B}{A}$.

$\boxed{\text{II}}$ Damped Free Vibrations : $m\,u'' + \gamma\,u' + k\,u = 0$

(i) $\gamma^2 - 4km > 0$ (*overdamped*) $\Longleftrightarrow$ distinct real roots to CE

(ii) $\gamma^2 - 4km = 0$ (*critically damped*) $\Longleftrightarrow$ repeated roots to CE

(iii) $\gamma^2 - 4km < 0$ (*underdamped*) $\Longleftrightarrow$ complex roots to CE (motion is *oscillatory*)

$\boxed{\text{III}}$ Forced Vibrations : ($F(t) = F_0\cos\omega t$ or $F(t) = F_0\sin\omega t$, for example)

(i) $m\,u'' + \gamma\,u' + k\,u = F(t)$ (Damped) In this case if you write the general solution as $u(t) = u_T(t) + u_\infty(t)$, then $u_T(t) =$ *Transient Solution* (i.e. the part of $u(t)$ such that $u_T(t) \longrightarrow 0$ as $t \longrightarrow \infty$) and $u_\infty(t) =$ *Steady-State Solution* (the solution behaves like this function in the long run).

(ii) $m\,u'' + k\,u = F_0\cos\omega t$ (Undamped) If $\omega = \omega_0 = \sqrt{\dfrac{k}{m}} \Rightarrow$ Resonance occurs and the solution is unbounded; while if $\omega \neq \omega_0$ then motion is a series of *beats* (solution is bounded)

(7) $\mathbf{n^{th}}$ Order Linear Homogeneous Equations With Constant Coefficients

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0 \quad (*)$$

This differential equation has $\boldsymbol{n}$ independent solutions.

Characteristic Equation : $a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1}r + a_n = 0$ will have $\boldsymbol{n}$ characteristic roots that may be real and distinct, repeated, complex, or complex and repeated.

(a) For each real root r that is not repeated $\Rightarrow$ get a solution of $(*)$: e^{rt}

(b) For each real root r that is repeated $\underline{\boldsymbol{m}}$ times $\Rightarrow$ get $\underline{\boldsymbol{m}}$ independent solutions of $(*)$:

$$e^{rt},\ te^{rt},\ t^2e^{rt},\ \cdots,\ t^{m-1}e^{rt}$$

(c) For each complex root $r = \lambda + i\mu$ repeated $\underline{\boldsymbol{m}}$ times $\Rightarrow$ get $\underline{\boldsymbol{2m}}$ solutions of $(*)$:

$$e^{\lambda t}\cos\mu t,\ te^{\lambda t}\cos\mu t,\ \cdots,\ t^{m-1}e^{\lambda t}\cos\mu t \quad \underline{\text{and}} \quad e^{\lambda t}\sin\mu t,\ te^{\lambda t}\sin\mu t,\ \cdots,\ t^{m-1}e^{\lambda t}\sin\mu t$$

(don't need to consider its conjugate root $\lambda - i\mu$)

(8) Undetermined Coefficients for n^{th} Order Linear Equations

This can only be used to find $y_p(t)$ of $a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = g(t)$ and $g(t)$ one of the 3 very SPECIAL FORMS in table in **(5)** above. The particular solution has the same form as before : $y_p(t) = t^s[\cdots]$, where $s =$ the smallest nonnegative integer such that no term of $y_p(t)$ is a term of $y_c(t)$, except this time $s = 0, 1, 2, \ldots, n$.

(9) Laplace Transforms

(a) Be able to compute Laplace transforms using definition :

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt}$$

and using a table of Laplace transforms (see table on page 317) and using linearity : $\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$, $\mathcal{L}\{c\,f(t)\} = c\,\mathcal{L}\{f(t)\}$.

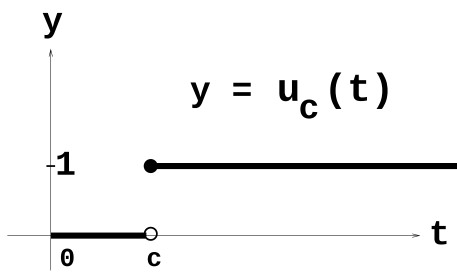
(b) Computing Inverse Laplace Transforms: Must be able to use a table of Laplace transforms usually together with *Partial Fractions* or *Completing the Square*, to find inverse Laplace transforms: $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

(c) Solving Initial Value Problems: Recall that

$$\boxed{\begin{aligned} &\mathcal{L}\{y'\} = s\,\mathcal{L}\{y\} - y(0) \\ &\mathcal{L}\{y''\} = s^2\,\mathcal{L}\{y\} - s\,y(0) - y'(0) \\ &\mathcal{L}\{y'''\} = s^3\,\mathcal{L}\{y\} - s^2\,y(0) - s\,y'(0) - y''(0) \\ &\qquad\vdots \end{aligned}}$$

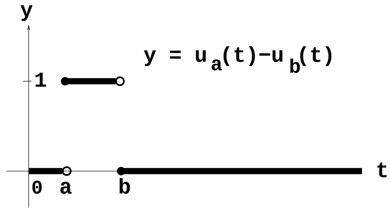
(d) Discontinuous Functions :

(i) **Unit Step Function** (Heaviside Function) : If $c \geq 0$, $u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$

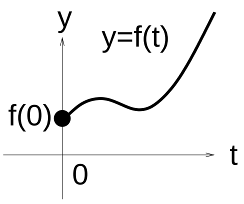


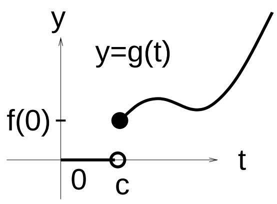
$$\boxed{\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}}$$

(ii) **Unit "Pulse" Function** : $u_a(t) - u_b(t) = \begin{cases} 1, & a \le t < b \\ 0, & \text{otherwise} \end{cases}$



(iii) **Translated Functions**: $y = g(t) = \begin{cases} 0\,, & t < c \\ f(t-c)\,, & t \ge c \end{cases} = u_c(t)\, f(t-c)\,.$





$$\boxed{\mathcal{L}\{u_c(t)\, f(t-c)\} = e^{-cs}\, F(s)\,, \text{where } F(s) = \mathcal{L}\{f(t)\}}$$

Thus, $$\boxed{\mathcal{L}^{-1}\{e^{-cs}\, F(s)\} = u_c(t)\, f(t-c)\,, \quad \text{where } f(t) = \mathcal{L}^{-1}\{F(s)\}}$$

A useful formula **NOT** in the book : $$\boxed{\boldsymbol{\mathcal{L}\{u_c(t)\, h(t)\} = e^{-cs}\, \mathcal{L}\{h(t+c)\}}}$$

(iv) **Unit Impulse Functions**: If $y = \delta(t-c)$ $(c \ge 0)$, then $$\boxed{\mathcal{L}\{\delta(t-c)\} = e^{-cs}}$$

(e) Convolutions: $$\boxed{\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\left\{\int_0^t f(t-\tau)\, g(\tau)\, d\tau\right\} = \mathcal{L}\{f(t)\}\ \mathcal{L}\{g(t)\}}$$

(10) Systems of Linear Differential Equations : $\mathbf{x}'(t) = \mathbf{A}\,\mathbf{x}(t)$

(a) Rewrite a single n^{th} order equation $p_0(t)y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = g(t)$ as a system of 1^{st} order equations. Use the substitution :

Let $\begin{array}{l} \mathtt{x}_1 = \mathtt{y} \\ \mathtt{x}_2 = \mathtt{y}' \\ \vdots \\ \mathtt{x}_n = \mathtt{y}^{(n-1)} \end{array}$ to get 1^{st} Order System : $\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \quad\vdots \\ x_{n-1}' = x_{n-2} \\ x_n' = \dfrac{1}{p_0}\{-p_n x_1 - p_{n-1}x_2 - \cdots - p_1 x_n + g(t)\} \end{cases}$

(b) **Existence & Uniqueness Theorem for Systems**. If $\mathbf{P}(t)$ and $\mathbf{g}(t)$ are continuous on an interval $\alpha < t < \beta$ containing t_0, then the IVP $\begin{cases} \mathbf{x}'(t) = \mathbf{P}(t)\,\mathbf{x}(t) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$ has a unique solution $\mathbf{x}(t)$ defined on the interval $\alpha < t < \beta$.

(c) The set of vectors $\left\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(m)}\right\}$ is *linearly independent* if the equation

$$k_1\mathbf{x}^{(1)} + k_2\mathbf{x}^{(2)} + \cdots + k_m\mathbf{x}^{(m)} = \mathbf{0}$$

is satisfied only for $k_1 = k_2 = \cdots = k_m = 0$. This means you cannot write any one of these vectors as a linear combination of the others.

(d) Solve 2×2 systems of 1^{st} order equations $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ i.e., $\mathbf{x}' = A\,\mathbf{x}$ using :

(i) **Elimination Method** : Basic idea - eliminate one of the unknowns (either x_1 or x_2) from the original system to get an equivalent single 2^{nd} order differential equation.

(ii) **Eigenvalues & Eigenvectors Method** : See **(11)** below for solutions via this method and corresponding phase portraits.

`Eigenvalue` : If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the eigenvalues of A are the roots of

$$|A - \lambda\, I| = \begin{vmatrix} (a-\lambda) & b \\ c & (d-\lambda) \end{vmatrix} = 0$$

`Eigenvector` : $\vec{\mathbf{v}} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a *nonzero* solution to $(A - \lambda\, I)\,\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

(e) If $\mathbf{x}^{(1)}(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \end{pmatrix}$ and if $\mathbf{x}^{(2)}(t) = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \end{pmatrix}$, then the *Wronskian* is

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}] = \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} .$$

If $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are solutions of $\mathbf{x}' = \mathbf{A}\,\mathbf{x}$ and $W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t_1) \neq 0$, then the set $\{\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t)\}$ forms a *Fundamental Set of Solutions* of the system and a *Fundamental Matrix* is $\Phi(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{pmatrix}$.

(11) Eigenvalue & Eigenvector Method and Phase Portraits : $\mathbf{x}' = \mathbf{A}\,\mathbf{x}$

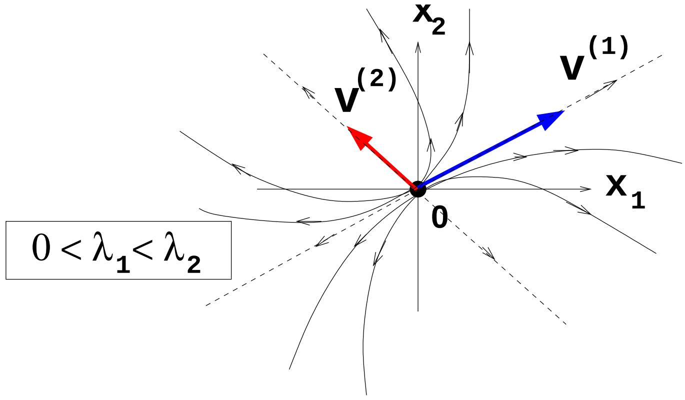
The following describes how to find the general solution to $(*)$ and plot solutions (trajectories). A plot of the trajectories of a given homogeneous system

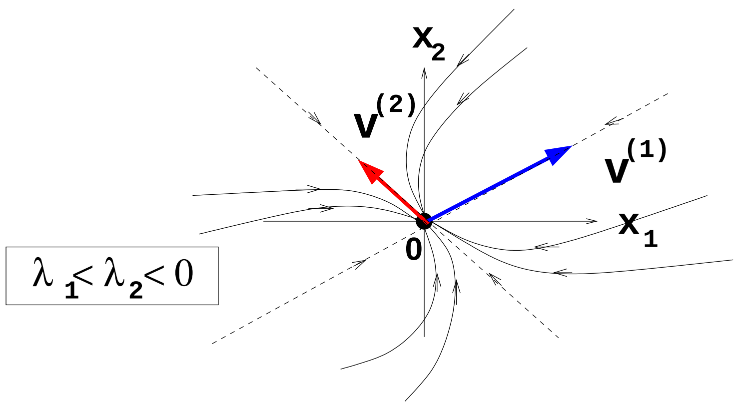
$$\mathbf{x}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x} \quad (*)$$

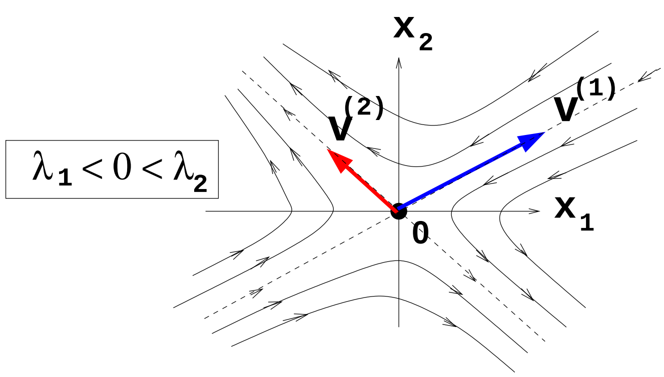
is called a **phase portrait**. To sketch the phase portrait, we need to find the corresponding eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and then consider 3 cases :

(a) $\underline{\boldsymbol{\lambda_1 < \lambda_2},\ \text{ real and distinct}}$: If $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$ are e-vectors corresponding to λ_1 and λ_2, respectively $\Rightarrow$ $\mathbf{x}^{(1)}(t) = e^{\lambda_1 t}\,\mathbf{v}^{(1)}$ and $\mathbf{x}^{(2)}(t) = e^{\lambda_2 t}\,\mathbf{v}^{(2)}$ are solutions and hence general solution of $(*)$ is $\mathbf{x}(t) = C_1\,\mathbf{x}^{(1)}(t) + C_2\,\mathbf{x}^{(2)}(t)$ and hence if $\boxed{\boldsymbol{\lambda_1 < \lambda_2}}$:

$$\mathbf{x}(t) = \underbrace{C_1\,e^{\lambda_1 t}\,\mathbf{v}^{(1)}}_{\substack{\texttt{dominates} \\ \texttt{as } t \longrightarrow -\infty}} + \underbrace{C_2\,e^{\lambda_2 t}\,\mathbf{v}^{(2)}}_{\substack{\texttt{dominates} \\ \texttt{as } t \longrightarrow \infty}}$$

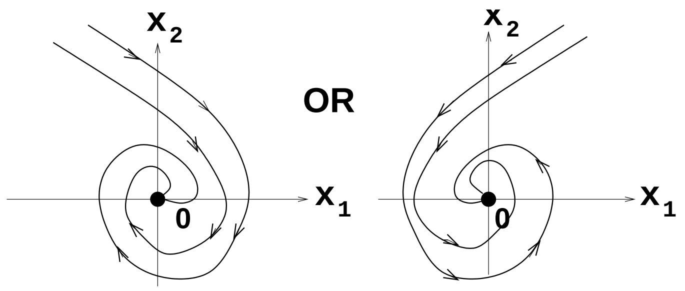






(b) $\underline{\boldsymbol{\lambda_1 = \alpha + i\,\beta}}$: If $\mathbf{w} = \mathbf{a} + i\,\mathbf{b}$ is a complex e-vector corresponding to λ_1 then $\Rightarrow$
$\mathbf{x}^{(1)}(t) = \Re e\left\{e^{\lambda_1 t}\mathbf{w}\right\} = e^{\alpha\,t}\,(\mathbf{a}\,\cos\beta t - \mathbf{b}\,\sin\beta t)$ and
$\mathbf{x}^{(2)}(t) = \Im m\left\{e^{\lambda_1 t}\mathbf{w}\right\} = e^{\alpha\,t}\,(\mathbf{a}\,\sin\beta t + \mathbf{b}\,\cos\beta t)$ are real-valued solutions and hence general solution of $(*)$ is $\mathbf{x}(t) = C_1\,\mathbf{x}^{(1)}(t) + C_2\,\mathbf{x}^{(2)}(t)$.

If say $\alpha < 0$:



(Test a point to decide which)

(c) $\underline{\boldsymbol{\lambda_1 = \lambda_2}}$: If there is only *one* linearly independent eigenvector corresponding to λ_1, then solutions to $\mathbf{x}' = \mathbf{A}\,\mathbf{x}$ are $\mathbf{x}^{(1)}(t) = e^{\lambda_1 t}\,\mathbf{v}$ and $\mathbf{x}^{(2)}(t) = t\,e^{\lambda_1 t}\,\mathbf{v} + e^{\lambda_1 t}\,\mathbf{a}$, where

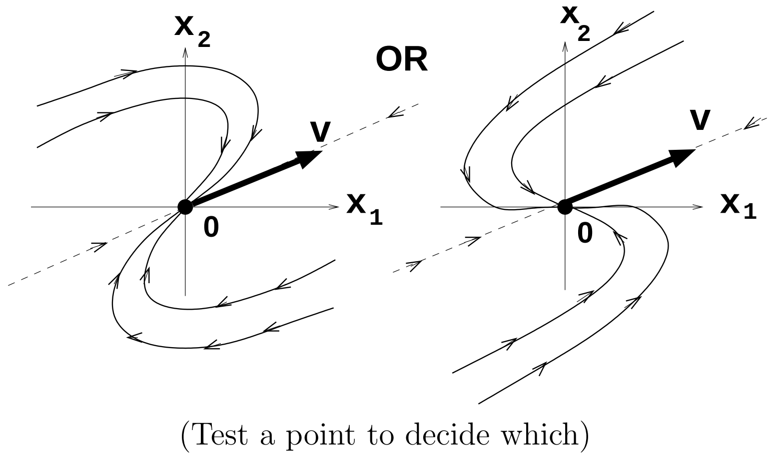
$$\begin{aligned}(A - \lambda_1\,I)\,\mathbf{v} &= \mathbf{0}\\ (A - \lambda_1\,I)\,\mathbf{a} &= \mathbf{v}\end{aligned}$$

($\mathbf{v}$ is an eigenvector of $\mathbf{A}$, while $\mathbf{a}$ is called a "generalized eigenvector" of $\mathbf{A}$)

The general solution of the system $(*)$ is $\mathbf{x}(t) = C_1\,\mathbf{x}^{(1)}(t) + C_2\,\mathbf{x}^{(2)}(t)$ and hence:

$$\mathbf{x}(t) = C_1\,e^{\lambda_1 t}\,\mathbf{v} + C_2\,\Big[\underbrace{\underline{t\,e^{\lambda_1 t}\,\mathbf{v}}}_{\substack{\texttt{dominates}\\ \texttt{as } t \longrightarrow \pm\infty}} + e^{\lambda_1 t}\,\mathbf{a}\Big]$$

If say $\lambda_1 < 0$:



(Test a point to decide which)

(12) Particular Solutions to Nonhomogeneous Linear Systems :

$$\mathbf{x}' = \mathbf{A}\,\mathbf{x} + \mathbf{g}(t)$$

(a) **Undetermined Coefficients for Systems** The column vector $\vec{\mathbf{g}}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$ must have each component function $g_1(t)$ and $g_2(t)$ as one of the three special forms like those for Undetermined Coefficients for regular 2^{nd} order equations and $\mathbf{A}$ must be a constant matrix. The main difference is if say $\mathbf{g}(t) = \mathbf{u}\,e^{\lambda t}$ and λ is also an eigenvalue of $\mathbf{A}$, then try a particular solution of the form $\mathbf{x}_p = \mathbf{a}\,te^{\lambda t} + \mathbf{b}\,e^{\lambda t}$.

(b) **Variation of Parameters for Systems** : $\mathbf{x}' = \mathbf{A}(\mathbf{t})\,\mathbf{x} + \mathbf{g}(t)$:

$$\mathbf{x}_p(t) = \Phi(t)\int \Phi^{-1}(t)\,\mathbf{g}(t)\,dt\,,$$

where $\Phi(t)$ is a Fundamental Matrix of the homogeneous system $\mathbf{x}' = \mathbf{A}(\mathbf{t})\,\mathbf{x}+\,\mathbf{g}(t)$ can have any form and $\mathbf{A}$ need not be a constant matrix.

Practice Problems

[**1**] For what value of α will the solution to the IVP $\begin{cases} y'' - y' - 2y = 0 \\ y(0) = \alpha \\ y'(0) = 2 \end{cases}$ satisfy $y \to 0$ as $t \to \infty$?

[**2**] (a) Show that $y_1 = x$ and $y_2 = x^{-1}$ are solutions of the differential equation $x^2y'' + xy' - y = 0$.
(b) Evaluate the Wronskian $W(y_2, y_1)$ at $x = \frac{1}{2}$.
(c) Find the solution of the initial value problem $x^2y'' + xy' - y = 0,\ y(1) = 2,\ y'(1) = 4$.

[**3**] Find the largest open interval for which the initial value problem
$3x^2y'' + y' + \dfrac{1}{x-2}y = \dfrac{1}{x-3},\ y(1) = 3,\ y'(1) = 2$, has a solution.

In Problems 4, 5, and 6 find the general solution of the homogeneous differential equations in (a) and use the method of **Undetermined Coefficients** to find a particular solution y_p in (b) and find the FORM of a particular solution (c).

[**4**] (a) $y'' - 5y' + 6y = 0$ (b) $y'' - 5y' + 6y = t^2$ (c) $y'' - 5y' + 6y = e^{2t} + \cos(3t)$

[**5**] (a) $y'' - 6y' + 9y = 0$ (b) $y'' - 6y' + 9y = te^{3t}$ (c) $y'' - 6y' + 9y = e^{t} + \cos(3t)$

[**6**] (a) $y'' - 2y' + 10y = 0$ (b) $y'' - 2y' + 10y = e^{x} + \cos(3x)$ (c) $y'' - 2y' + 10y = e^{x}\cos(3x)$

[**7**] Find the general solution to (a) $y'' + y' - 6y = 7e^{4t}$ (b) $y'' + y' - 6y = 7e^{4t} - 100\sin t$

[**8**] Solve this IVP: $y'' - y' = 4t,\ y(0) = 0,\ y'(0) = 0$.

[**9**] Find the general solution to $y'' + y = \tan t\,,\ 0 < x < \frac{\pi}{2}$.

[**10**] The differential equation $x^2y'' - 2xy' + 2y = 0$ has solutions $y_1(x) = x$ and $y_2 = x^2$. Use the method of **Variation of Parameters** to find a solution of $x^2y'' - 2xy' + 2y = 2x^2$.

[**11**] The differential equation $x^2y''+xy'-y = 0$ has one solution $y_1(x) = x$. Use the method of **Reduction of Order** to find a second (linearly independent) solution of $x^2y'' + xy' - y = 0$.

[**12**] For what nonnegative values of γ will the the solution of the initial value problem
$u'' + \gamma u' + 4u = 0,\ u(0) = 4,\ u'(0) = 0$ *oscillate* ?

[**13**] (a) For what positive values of k does the solution of the initial value problem
$2u'' + ku = 3\cos(2t),\ u(0) = 0,\ u'(0) = 0,$ become *unbounded* (Resonance) ?

(b) For what positive values of k does the solution of the initial value problem
$2u'' + u' + ku = 3\cos(2t),\ u(0) = 0,\ u'(0) = 0,$ become *unbounded* (Resonance) ?

[14] Find the steady–state solution of the IVP $y'' + 4y' + 4y = \sin t,\ y(0) = 0,\ y'(0) = 0.$

[15] A 4-kg mass stretches a spring 0.392 m. If the mass is released from 1 m below the equilibrium position with a downward velocity of 10 m/sec, what is the maximum displacement ?

In Problems 16 and 17 find the general solution of the homogeneous differential equations in (a) and use the method of **Undetermined Coefficients** to find the <u>FORM</u> of a particular solution of the nonhomogeneous equation in (b).

[16] (a) $y''' - y' = 0$ (b) $y''' - y' = t + e^t$

[17] (a) $y''' - y'' - y' + y = 0$ (b) $y''' - y'' - y' + y = e^t + \cos t$

[18] Find the solution of the initial value problem $y''' - 2y'' + y' = 0,\ y(0) = 2,\ y'(0) = 0,\ y''(0) = 1.$

[19] Find the general solution of the differential equation $y''' + y' = t^2$.

[20] Find the general solution of $y'' + 4y' = -10\cos 2t$.

[21] Find a fundamental set of solutions of $y^{(5)} - 4y''' = 0$.

[22] Find the Laplace transform of these functions:
(a) $f(t) = 3 - e^{2t}$ (b) $g(t) = 100\,t^5$ (c) $h(t) = \cosh \pi t$ (d) $k(t) = -10t^3 e^{5t}$

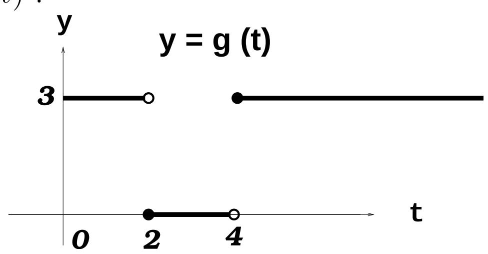
[23] Find the inverse Laplace transform of
(a) $F(s) = \dfrac{9}{s^2 - s - 2}$ (b) $F(s) = \dfrac{s}{(s-1)^2}$ (c) $F(s) = \dfrac{8}{(s+1)^4}$ (d) $F(s) = \dfrac{3s+2}{s^2 + 2s + 5}$

[24] Solve these initial value problems: (a) $\begin{cases} y'' - y' - 6y = 0 \\ y(0) = 1 \\ y'(0) = -1 \end{cases}$ (b) $\begin{cases} y'' - 2y' + 2y = \cos t \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$

(c) $y'' - y = \begin{cases} 1, & t < 5 \\ 2, & 5 \le t < \infty \end{cases}$; $y(0) = y'(0) = 0.$

(d) $y'' + 4y = \begin{cases} t, & t < 1 \\ 0, & 1 < t < \infty \end{cases}$; $y(0) = y'(0) = 0.$

(e) $y' + y = g(t)$, $y(0) = 0$ and where $g(t)$:



(f) $y'' + 4y = \delta(t-3)$, $y(0) = y'(0) = 0$

[25] $\mathcal{L}\left\{\int_0^t 100\,e^{-2\tau}\cos\pi(t-\tau)\,d\tau\right\} = ?$

[26] If $g(t) = \mathcal{L}^{-1}\{G(s)\}$, then $\mathcal{L}^{-1}\left\{\dfrac{G(s)}{(s-3)^2}\right\} = ?$

[27] Use the <u>Elimination Method</u> to solve the system $\begin{cases} x_1' = x_1 + x_2 \\ x_2' = 4x_1 + x_2 \end{cases}$

[28] Rewrite the 2^{nd} order differential equation $y'' + 2y' + 3ty = \cos t$ with $y(0) = 1, y'(0) = 4$ as a system of 1^{st} order differential equations.

[29] Find eigenvalues and corresponding eigenvectors of (a) $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ (b) $A = \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix}$

[**30**] Find the solution of the IVP $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$

Find a fundamental matrix $\Phi(t)$.

[**31**] Solve $\begin{pmatrix} x_1{}' \\ x_2{}' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \vec{\mathbf{x}}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$

[**32**] Find the general solution of the system $\vec{\mathbf{x}}\,'(t) = A\,\vec{\mathbf{x}}(t)$, where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$

[**33**] Tank # 1 initially holds 50 gals of brine with a concentration of 1 lb/gal, while Tank # 2 initially holds 25 gals of brine with a concentration of 3 lb/gal. Pure H_2O flows into Tank # 1 at 5 gal/min. The well-stirred solution from Tank # 1 then flows into Tank # 2 at 5 gal/min . The solution in Tank # 2 flows out at 5 gal/min. Set up and solve an IVP that gives $x_1(t)$ and $x_2(t)$, the amount of salt in Tanks # 1 and # 2, respectively, at time t.

[**34**] Tank # 1 initially holds 50 gals of brine with concentration of 1 lb/gal and Tank # 2 initially holds 25 gals of brine with concentration 3 lb/gal. The solution in Tank # 1 flows at 5 gal/min into Tank # 2, while the solution in Tank # 2 flows back into Tank # 1 at 5 gal/min. Set up an IVP that gives $x_1(t)$ and $x_2(t)$, the amount of salt in Tanks # 1 and # 2, respectively, at time t.

[**35**] Find the general solution of $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^t.$

[**36**] Find a particular solution of $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$

[**37**] Find the general solution of $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 6e^{-t} \\ 1 \end{pmatrix}.$

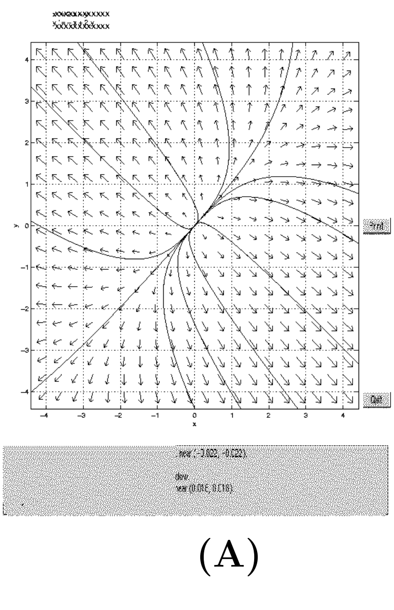
[**38**] Match the phase portraits shown below that best corresponds to each of the given systems of differential equations:

(i) $\vec{\mathbf{x}}\,' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{\mathbf{x}}$; Solution : $\vec{\mathbf{x}}(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$

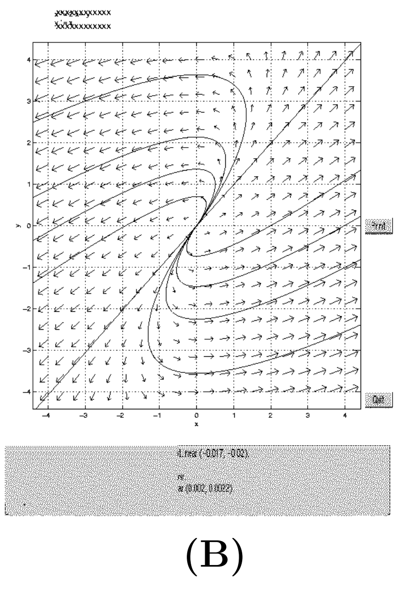
(ii) $\vec{\mathbf{x}}\,' = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \vec{\mathbf{x}}$; Solution : $\vec{\mathbf{x}}(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t}$

(iii) $\vec{\mathbf{x}}\,' = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \vec{\mathbf{x}}$; Solution : $\vec{\mathbf{x}}(t) = C_1 \begin{pmatrix} 2 \\ 2 \end{pmatrix} e^t + C_2 e^t \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix} t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

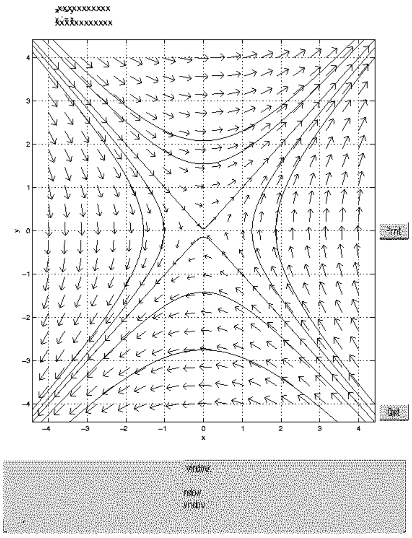
(iv) $\vec{\mathbf{x}}\,' = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \vec{\mathbf{x}}$; Solution : $\vec{\mathbf{x}}(t) = C_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} e^{-t}$



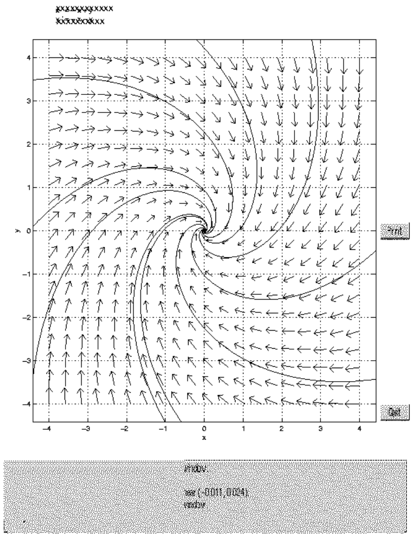
(A)



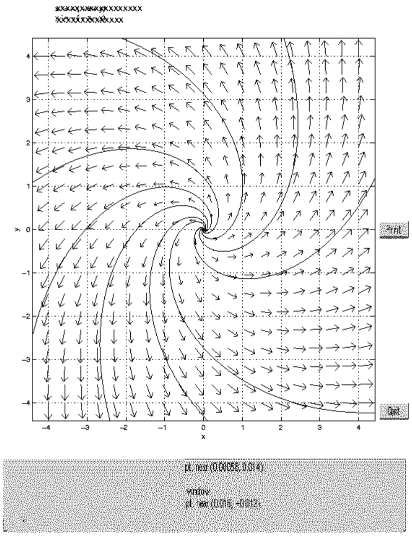
(B)



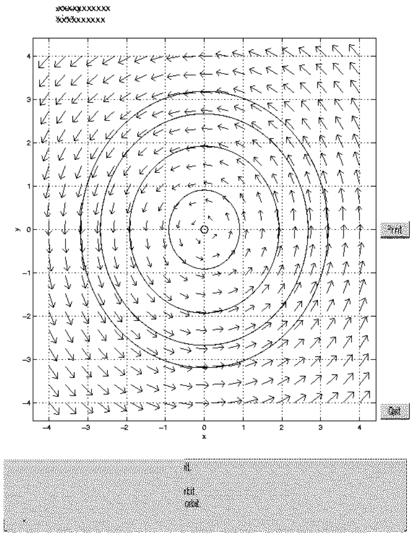
(C)



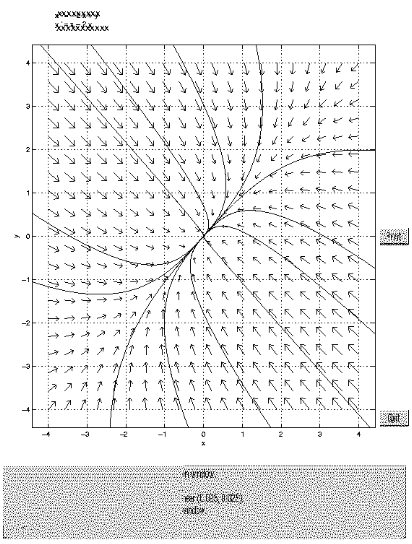
(D)



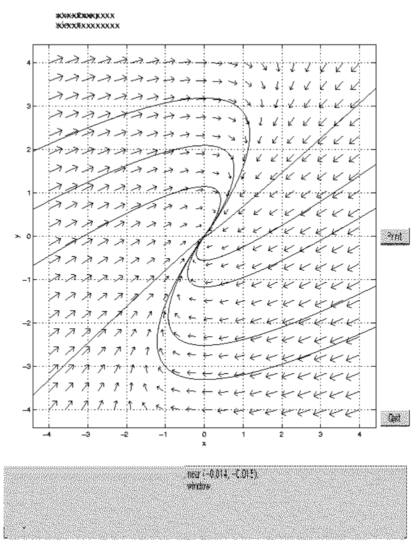
(E)



(F)



(G)



(H)

Answers

[**1**] $\alpha = -2$ [**2**] (b) $W(x^{-1}, x)(\frac{1}{2}) = 4$; (c) $y = 3x - x^{-1}$ [**3**] $0 < x < 2$
[**4**] (a) $y = C_1 e^{2t} + C_2 e^{3t}$ (b) $y = At^2 + Bt + C$ (c) $y = Ate^{2t} + B\cos(3t) + C\sin(3t)$
[**5**] (a) $y = C_1 e^{3t} + C_2 te^{3t}$ (b) $y = t^2(At + B)e^{3t}$ (c) $y = Ae^t + B\cos(3t) + C\sin(3t)$
[**6**] (a) $y = C_1 e^x \cos(3x) + C_2 e^x \sin(3x)$ (b) $y = Ae^x + B\cos(3x) + C\sin(3x)$
(c) $y = x(A\cos(3x) + B\sin(3x))e^x$
[**7**](a) $y = C_1 e^{-3t} + C_2 e^{2t} + \frac{1}{2}e^{4t}$ (b) $y = C_1 e^{-3t} + C_2 e^{2t} + \frac{1}{2}e^{4t} + 2\cos t + 14\sin t$
[**8**] $y = -4 + 4e^t - 2t^2 - 4t$
[**9**] $y = C_1 \cos t + C_2 \sin t - (\cos t)\ln(\sec t + \tan t)$
[**10**] $y = 2x^2 \ln x$ or $y = 2x^2 \ln x + (C_1 x + C_2 x^2)$
[**11**] $y = x^{-1}$ or $y = Ax^{-1} + Bx,\ A \neq 0$
[**12**] $0 \le \gamma < 4$
[**13**] (a) $k = 8$ (resonance) (b) NO value of k, all solutions are bounded.
[**14**] $y = \frac{1}{25}(3\sin t - 4\cos t)$
[**15**] $u(t) = \cos 5t + 2\sin 5t = \sqrt{5}\cos(5t - \delta),\ \delta = \tan^{-1} 2 \approx 1.1$ Thus amplitude $= \sqrt{5}$.
[**16**] (a) $y = C_1 + C_2 e^{-t} + C_3 e^t$ (b) $y = t(At + B) + Cte^t$
[**17**] (a) $y = C_1 e^t + C_2 te^t + C_3 e^{-t}$ (b) $y = At^2 e^t + B\cos t + C\sin t$
[**18**] $y = 3 - e^t + te^t$
[**19**] $y = C_1 + C_2 \cos t + C_3 \sin t + \dfrac{1}{3}t^3 - 2t$
[**20**] $y = C_1 + C_2 e^{-4t} + \left(\frac{1}{2}\cos 2t - \sin 2t\right)$
[**21**] $\{1, t, t^2, e^{2t}, e^{-2t}\}$
[**22**] (a) $\dfrac{2s-6}{s^2-2s}$ (b) $\dfrac{12000}{s^6}$ (c) $\dfrac{s}{s^2-\pi^2}$ (d) $-\dfrac{60}{(s-5)^4}$
[**23**] (a) $3(e^{2t} - e^{-t})$ (b) $e^t + te^t$ (c) $\frac{4}{3}t^3 e^{-t}$ (d) $3e^{-t}\cos 2t - \frac{1}{2}\,e^{-t}\sin 2t$
[**24**] (a) $y = \frac{1}{5}(e^{3t} + 4e^{-2t})$ (b) $y = \frac{1}{5}(\cos t - 2\sin t + 4e^t \cos t - 2e^t \sin t)$
(c) $y = -1 + \frac{1}{2}(e^t + e^{-t}) + u_5(t)(-1 + \frac{1}{2}(e^{(t-5)} + e^{-(t-5)}))$,
or $y = -1 + \cosh t + u_5(t)(-1 + \cosh(t-5))$
(d) $y = (-\frac{1}{8}\sin 2t + \frac{t}{4}) - u_1(t)(-\frac{1}{8}\sin 2(t-1) + \frac{t-1}{4}) - u_1(t)(\frac{1}{4} - \frac{1}{4}\cos 2(t-1))$
(e) $y = 3(1 - e^{-t}) - 3\,u_2(t)(1 - e^{-(t-2)}) + 3\,u_4(t)(1 - e^{-(t-4)})$ (f) $y = \frac{1}{2}u_3(t)(t)\sin 2(t-3)$

[**25**] $\dfrac{100\,s}{(s+2)(s^2+\pi^2)}$ [**26**] $\displaystyle\int_0^t (t-\tau)\,e^{3(t-\tau)} g(\tau)\,d\tau$ or $\displaystyle\int_0^t \tau\,e^{3\tau} g(t-\tau)\,d\tau$

[**27**] $x_1(t) = C_1 e^{3t} + C_2 e^{-t},\ x_2(t) = 2C_1 e^{3t} - 2C_2 e^{-t}$

[**28**] Let $x_1 = y,\ x_2 = y'$, then $\begin{cases} x_1' = x_2 \\ x_2' = -3tx_1 - 2x_2 + \cos t \end{cases}$, where $x_1(0) = 1,\ x_2(0) = 4$

[**29**] (a) $\lambda_1 = 3,\ \mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $\lambda_2 = -1,\ \mathbf{v}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

[**29**] (b) $\lambda_1 = -1,\ \mathbf{v}^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $\lambda_2 = -2,\ \mathbf{v}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

[**30**] $\mathbf{x}(t) = 2\,e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $\Phi(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$

[**31**] $\mathbf{x}(t) = 2\,e^t\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} - e^t\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$ [**32**] $\mathbf{x}(t) = C_1 e^t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2\left\{e^t\begin{pmatrix} 0 \\ 1 \end{pmatrix} + te^t\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$

[**33**] $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -\frac{1}{10} & 0 \\ \frac{1}{10} & -\frac{1}{5} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 50 \\ 75 \end{pmatrix}$

Solution : $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 50e^{-\frac{t}{10}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 25e^{-\frac{t}{5}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

[34] $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -\frac{1}{10} & \frac{1}{5} \\ \frac{1}{10} & -\frac{1}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 50 \\ 75 \end{pmatrix}$

Solution : $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{125}{3} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{100}{3} e^{-\frac{3t}{10}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

[35] $\mathbf{x}(t) = C_1 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ **[36]** $\mathbf{x}_p(t) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

[37] $\mathbf{x}(t) = C_1 e^{t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

[38] (i) **C** (ii) **A** (iii) **B** (iv) **D**

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$
2.	e^{at}	$\frac{1}{s-a}$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	$t^p \ (p > -1)$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sin at$	$\frac{a}{s^2+a^2}$
6.	$\cos at$	$\frac{s}{s^2+a^2}$
7.	$\sinh at$	$\frac{a}{s^2-a^2}$
8.	$\cosh at$	$\frac{s}{s^2-a^2}$
9.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
10.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
11.	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
11.	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
12.	$u_c(t)$	$\frac{e^{-cs}}{s}$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \ c > 0$
16.	$\int_0^t f(t-\tau)\,g(\tau)\,d\tau$	$F(s)\,G(s)$
17.	$\delta(t-c)$	e^{-cs}
18.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$
19.	$(-t)^n f(t)$	$F^{(n)}(s)$